Market Clearing with Semi-fungible Assets

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Abstract

As markets have digitized, the number of tradable products has skyrocketed. Algorithmically constructed portfolios of these assets now dominate public and private markets, resulting in a combinatorial explosion of tradable assets. In this paper, we provide a simple means to compute market clearing prices for *semi-fungible* assets which have a partial ordering between them. Such assets are increasingly found in traditional markets (bonds, commodities, ETFs), private markets (private credit, compute markets), and in decentralized finance. We formulate the market clearing problem as an optimization problem over a directed acyclic graph that represents participant preferences. Subsequently, we use convex duality to efficiently estimate market clearing prices, which correspond to particular dual variables. We then describe dominant strategy incentive compatible payment and allocation rules for clearing these markets. We conclude with examples of how this framework can construct prices for a variety of algorithmically constructed, semi-fungible portfolios of practical importance.

1 Introduction

Many electronically traded assets are non-fungible. In contrast to a company's common stock, which is all—by construction—fungible, many other assets are similar but not exactly identical. For a bond, we might be interested in its yield and credit rating. For a metal, we may care about not only the price at which it is traded but also the purity level and delivery location. For a liquid staking token, we might also care about the reputation of the staking service providers. In all these examples, the items cannot be totally ordered—in contrast to a company's stock, which can be ordered by price in some order book. Instead, these items are partially ordered: some are better than others, some are worse than others, and some are incomparable. We call these types of assets *semi-fungible*. In this paper, we develop a market clearing model for divisible, semi-fungible assets, such as those tokenized on public blockchains.

Partial orders and embeddings. Partial orders are one natural interpolation between complete fungibility and non-fungibility. In particular, fungible assets can generally be easily ordered based on their price; a buyer always prefers a lower price. For semi-fungible assets, the asset price is only one of potentially many relevant dimensions. These assets naturally have some partial orders where some assets may be preferred to other assets and be further incomparable to others. As we will see, examples of semi-fungible assets range from bonds and commodities to online ad placement and some cryptocurrencies.

1.1 Examples

We motivate our setting with several examples that also provide helpful mental models to ground the mechanisms we subsequently develop.

Bonds. Bond traders historically considered each bond individually. More recently, systemic credit funds and portfolio trading have increasingly taken over as electronic trading has become more prevalent on these markets. The assets under management of these funds has more than doubled in the last year [New24], and portfolio trading—where large baskets of bonds are traded together—has similarly doubled since the beginning of 2024 [Whi24]. These developments are a result of the semi-fungibility of bonds: bonds with the same yield and the same credit rating are, for many purposes, equivalent. In the words of Matt Levine [Lev24]:

> Historically, though, for a long time there was a view that each bond is a special snowflake and you had to travel to the ends of the earth to get exactly the BBB+ bond that you were looking for. [...] The modern view is that bonds are largely linear combinations of factor exposures so liquidity is just fine.

In other words, these assets have a number of properties that can be partially ordered.

Semi-fungible commodities. Many financial products in commodities markets—even those traded on exchanges—possess a certain number of differentiating properties. For example, a given lot of a particular metal, such as nickel, may have specific purity levels, delivery locations, or some notion of seller reputation. Buyers may require a certain minimum level or purity from this lot, or may only trust a subset of sellers in a given market.

Compute markets. For a given compute workload, many different types of compute hardware may be available. For example, a machine learning model could be trained on the latest GPU (likely the fastest), on an older GPU (likely at a lower throughput), or on another type of hardware (*e.g.*, a CPU or a TPU). Buyers may have different preferences for the hardware they would like to use and the relative price-performance tradeoffs they are willing to make, but ultimately all buyers are purchasing FLOPs for some time period.

Liquid (re)staking tokens. Issuers of yield-bearing assets, such as treasuries or bonds, pay buyers some rate of interest over time. In return, the buyer accepts the risk that the seller may default on their payments. As a result, buyers often have a preference on the reputation of the issuers. In traditional markets, this reputation is typically assessed by rating agencies such as S&P or Moody's. A similar thing has played out in blockchain systems such as Ethereum and Solana. These systems have recently witnessed the rise of so-called liquid staking tokens, issued by staking pools such as Lido and Jito, which promise a yield on their staked assets. The recent rise of re-staking protocols (*e.g.*, Eigenlayer and Symbiotic) resulted in a proliferation of these yield-bearing assets. Each of these assets can differ on many dimensions: the yield rate, the reputation (and therefore risk) of the (re)-staking protocol, the perceived risk of default (via slashing), and the redemption mechanism, among many other axes.

Collectibles. Finally, collectible markets, such as art markets, sneaker markets, or NFT markets, exhibit similar partially-ordered properties. For example, a buyer may want a piece by a particular artist, or a piece from a particular collection, but may not care about the specifics of the piece beyond these properties. In NFT markets, this type of preference is common, evidenced by the fact that a large proportion of the collection usually trades close to the floor price—the lowest available price of any item in the collection.

1.2 Related work

We consider the market clearing problem with a partial ordering induced by divisible, semi-fungible assets. Our work builds on ideas that have been previously developed in the literature. We note that, because we consider divisible goods, our work differs from the literature on general combinatorial auctions.

Partially-ordered assets. Partially-ordered assets have been considered in the literature, often in the context of indivisible goods. This type of structure describes relationships for many types of assets: options on stocks can be (partially) ordered by their strike price and/or maturity [AD03]; bonds can be ordered by their rating [JLT97]; other debt products can be ordered by their seniority [LR08]; airline seats can be ordered by their class [BM93]; and so on. Other work has considered partially ordered items in more generality, for example, see [AR22]. Most of these works are not concerned with market clearing.

Batched exchanges. Our market mechanism works by batching together the preferences of many buyers and clearing the market at regular intervals, akin to the frequent batched auctions proposed by [BCS15]. We require the buyer preferences to be utility functions, which may be defined over portfolios of assets. This structure is similar to the bidding language developed by [Bud+23], which generalizes the orders from [KL17]. In contrast to this work, our work assumes the existance of and uses a partial order of preferences over the assets.

Market clearing. The broader literature on market clearing mechanisms is vast. Eisenberg and Gale proved that, for a market with divisible goods and buyers with linear utilities, the market equilibrium conditions are the optimality conditions of a particular convex program [EG59]. The case with concave utilities maps to a convex optimization problem [Vég14]. Many results in the literature have been developed for markets fully fungible or non-fungible goods using these models frameworks. (See [Vaz07; CV07] and citations therein.) A smaller set of work has considered market clearing when participants have preferences over the items. (See, for example, [CG11; GS99].) We focus on the specific case where there exists some global partial ordering over the items in this market, and buyers can express their preferences over the items in terms of this ordering.

1.3 This paper

In this paper, we introduce a market clearing model for divisible, semi-fungible assets by leveraging partial order relationships among different properties. We introduce a framework that captures partially ordered preferences across items in §2. Next, we define a convex optimization formulation for market clearing in §3. In §4, we derive market clearing prices and also propose a dominant strategy incentive compatible mechanism for clearing these markets. We conclude with both toy and real-world examples in §5.

2 The asset

We consider a single asset (*e.g.*, some particular metal, such as nickel) with some number of *properties*. For example, in metals markets, these properties may include the metal purity levels, delivery locations, or some notion of seller reputation. We denote the set of properties (which is some abstract set) as $\mathcal{P}$. We will refer to an *item* i as a (generally divisible) good with property p_i that can be purchased in some quantity.

Ordering over properties. We assume that these properties are *partially ordered*. That is, given two items with some properties $p, p' \in \mathcal{P}$, respectively, we have one of three cases: either $p \leq p'$; or $p \geq p'$; or, if neither of these are true, then p is *incomparable* to p'. The first, $p \leq p'$, may be read as or 'p is no better than p'', and similarly for the second, after interchanging p and p'. In addition, if there is a third item with properties $p'' \in \mathcal{P}$ which satisfy $p'' \geq p'$ then, if $p' \geq p$, we should have $p'' \geq p$; *i.e.*, the relation is transitive. Finally, we assume that the set of orders has the property that, if $p \geq p'$ and $p \leq p'$, then $p = p'$; *i.e.*, the relation is antisymmetric.

Interpretations. The ordering over items has the following interpretation. If we have two items i and i' with properties p_i and $p_{i'}$, respectively, then, if $p_i \leq p_{i'}$, we assume that, all else being equal, anyone would accept the item with properties $p_{i'}$ if they would accept the items with properties p_i. For example, given two metals, one with higher purity than the other, a buyer attempting to purchase the lower purity metal would also happily accept the higher purity metal at the same price, all else being equal. Similarly, a bond buyer would happily accept a higher yield, all else being equal. The case with $p_i \geq p_{i'}$ is similar, replacing the role of i with i'. Finally the case where p_i is incomparable to $p_{i'}$ means that some buyers might prefer item i over i' and some others might prefer i' over i. For example, if some metal originates from the United States versus the Philippines, some buyers might prefer the former over the latter and vice versa. Similarly, if some bond has a duration of one year and another has a duration of two years, some buyers might prefer the former over the latter and vice versa.

Discussion. We will show that it is possible to construct markets that make use of this additional partial-ordering structure in order to clear. These markets have a number of important guarantees about how orders are executed. For example, in this market, if a buyer expresses that she wants an item with properties at least as good as $p' \in \mathcal{P}$, then, if no such item is available, she can be matched with any item i that has properties $p_i \geq p'$, assuming such a item exists and is provided at no more than the price she is willing to pay. (In economics parlance, we say that she has access to 'more liquidity' in this market than a market that only trades items with properties p_i.)

Partial orders as DAGs. Any partial order with a finite number of elements can be viewed as a directed acyclic graph (often abbreviated to DAG) with $P = |\mathcal{P}|$ vertices, which we will label with the elements of $\mathcal{P}$. The edges are defined in the following way: the edge from $p \in \mathcal{P}$ to $p' \in \mathcal{P}$ is present if $p \leq p'$ and there is no $p'' \in \mathcal{P}$ such that $p \leq p'' \leq p'$. In other words, two nodes have an edge between them if there is no element that lies in between them (with respect to the ordering of $\mathcal{P}$). We will use this representation in what follows. A simple example for yield-bearing assets is shown in figure 1.

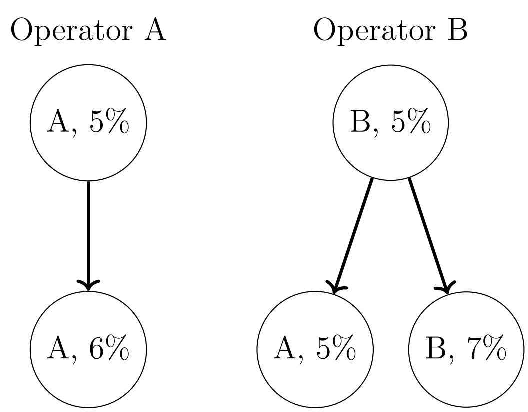

Figure 1: An example partial ordering on five yield-bearing assets with different ratings and yields, provided by different operators. We assume the operators are not comparable, whereas any buyer prefers a higher rating and higher yield, all else being equal.

2.1 The embedding

Here, we discuss how to express buyer preferences for the items.

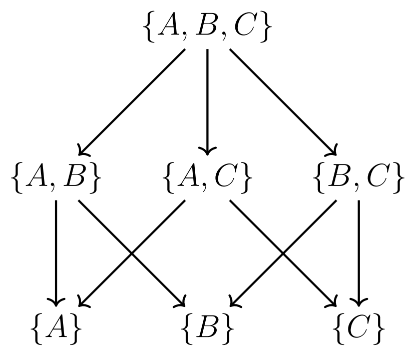

Figure 2: Partial orders on the set of, say, node operators. Some users may be indifferent between the operators A and B, while others may be indifferent between A and C, and so on.

Embedding the ordering. A buyer b can express her preferences for the asset by specifying a property $p_b \in \mathcal{P}$ which will, in some sense, be the base item she is willing to purchase. In other words, given a list of n items $i = 1, \ldots, n$ with properties $\{p_i\}$, she will take any item i satisfying $p_i \geq p_b$.

Preference vector. Let the vector $z_b \in \mathbf{R}^n_+$ denote the amount of each of the n items allocated to buyer b. (Buyer b need not want all of these items.) We then write $x_b \in \mathbf{R}_+$ for the total amount of the asset (which includes any items with properties at least as good as p_b) that buyer b is willing to buy. Written out in math, this is

$$x_b = \sum_{i \,:\, p_i \geq p_b} (z_b)_i.$$

The sum is taken over all items i such that with properties p_i are at least as good as buyer b's desired property; *i.e.*, the possible items' properties satisfy $p_i \geq p_b$. This can be written much more succinctly as

$$x_b = c_b^T z,$$

where we call the vector $c_b \in \{0, 1\}^n$ the *preference vector* for buyer b, defined

$$c_b = \sum_{i \,:\, p_i \geq p_b} e_i.$$

(Here, e_i it the ith unit basis vector.) More generally, we may wish to express weighted preferences between the categories. For example, if a buyer has access to a zero-cost purification process, she may be indifferent between one unit of an 100% pure metal and two units of a 50% pure metal. We can express this preference using a vector $c \in [0, 1]^n$, such that the 'total amount' of the asset received is $c^T z$. We leave these more general, non-Boolean preference vectors as an extension.

Preference vector ordering. Note that the preference vectors of buyers are themselves also ordered in the following way: for any buyers b and b', we have

$$p_b \le p_{b'} \quad \text{if, and only if,} \quad c_b \ge c_{b'},$$

where the first inequality is with respect to the partial ordering over $\mathcal{P}$ and the second is an elementwise inequality. Indeed, letting

$$\mathbf{supp}(z) = \{i \mid z_i \neq 0\},$$

be the *support* of a vector z; *i.e.*, the set of indices for which z is nonzero, then this is equivalent to saying that

$$p_b \le p_{b'} \quad \text{if, and only if,} \quad \mathbf{supp}(c_b) \supseteq \mathbf{supp}(c_{b'}), \tag{1}$$

since c_b and $c_{b'}$ are Boolean. In other words, buyers willing to accept items with worse properties have more permissive preference vectors. We make use of this relationship in what follows.

3 The exchange

We now specify the market clearing problem for partially ordered assets, as described above. We consider a single asset with B buyers, labeled $b = 1, \dots, B$ and one supplier with a quantity $q \in \mathbf{R}^n_+$ of each item to sell. The ith entry of q, denoted q_i, represents the total quantity of item i (with property p_i) that the supplier is willing to sell. Each buyer b receives some amount $x_b \in \mathbf{R}_+$ of the asset after the market clears.

Preferences. As discussed above, we use a vector $c_b \in \mathbf{R}^n$ to represent the preferences of a buyer over the items. If the buyer expresses that she wants items with properties $p' \in \mathcal{P}$, then, if no such item is available, she can be matched with any item i that has properties $p_i \ge p'$. Thus, $(c_b)_i = 1$ for all i such that $p_i \ge p'$ and $(c_b)_i = 0$ otherwise.

Allocations. We define vectors $z_b \in \mathbf{R}^n_+$ such that the ith entry of z_b, denoted $(z_b)_i$, is the amount of the item with property p_i that the supplier sells to buyer b. The total amount of the asset received by buyer b (that this buyer is willing to take) can be written as

$$x_b = c_b^T z_b.$$

Finally, we include a constraint ensuring the supplier does not sell more than the available items:

$$\sum_{b=1}^{B} z_b \le q.$$

Utilities. We collect the amount of the asset received by each buyer into a vector $x = (x_1, \dots, x_B) \in \mathbf{R}^B_+$. We express the utility of this allocation by a concave, nondecreasing function $U : \mathbf{R}^B_+ \to \mathbf{R} \cup \{-\infty\}$, where infinite values encode constraints: an allocation x such that $U(x) = -\infty$ is unacceptable. We also require that $U(0) = 0$ and assume that U is strictly increasing at 0, *i.e.*, if U is differentiable at 0, that $\nabla U(0) > 0$. We note that U is typically separable over the buyers; each buyer individually specifies some utility function on their individual allocation, denoted $u_b : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$. In this case, we can write U as

$$U(x) = \sum_{b=1}^{B} u_b(x_b).$$

Market clearing. We write the market clearing problem as

$$\begin{array}{ll} \text{maximize} & U(x) \\ \text{subject to} & x_b = c_b^T z_b \\ & \sum_{b=1}^{B} z_b \leq q \\ & z_b \geq 0, \qquad b = 1, \dots, B. \end{array} \tag{2}$$

The variables are $x \in \mathbf{R}^B$ and $z_b \in \mathbf{R}^n$ for each buyer $b = 1, \dots, B$. The problem data are the items supplied to the market $g \in \mathbf{R}^n_+$, the preferences $c_b \in \mathbf{R}^n_+$, and the utility function U, which is often a sum of buyer utilities u_b. This problem is easily recognized as a convex optimization problem with linear constraints. In fact, when the utility function U is piecewise linear, this problem is simply a linear program. In both cases, the problem can be solved efficiently in practice. Note that this problem can also be written as a special case of a network flow problem in the framework of [DAE24] (*cf.*, the Fisher market problem introduced in [DAE24, §3.4]).

Discussion. This market model also extends to the common scenario where buyers may wish to purchase multiple 'baskets' of items, each corresponding to different preferences and utility functions. For example, a bond buyer may want to express preferences over higher-rated and lower-rated bonds separately. In this case, we can simply introduce two 'buyers' b and b' corresponding to these different preferences in our model. Since the utility function is concave, this model allows for preferences with complements. (This is the case, for example, if we value two units of items which have properties at least as good as both A and B more than we value one unit of items with properties at least as good as A and one unit of items at least as good as B, but not necessarily both.) Our model and the associated analysis can also be easily extended to the scenario in which we have more than one seller, each with some allocation of items, and to the scenario with more than one asset.

4 Market-clearing mechanism

Now that we have defined the market clearing problem (2), which is a convex optimization problem, we derive and examine the dual problem to determine how we should clear this market.

4.1 Market-clearing prices

We can write the Lagrangian of the market clearing problem as

$$L(x, z, \nu, \lambda) = U(x) + \sum_{b=1}^{B} \nu_b (c_b^T z_b - x_b) + \lambda^T \left(q - \sum_{b=1}^{B} z_b \right) - \sum_{b=1}^{B} I(z_b),$$

where the dual variables are $\nu \in \mathbf{R}^B$ and $\lambda \in \mathbf{R}_+^n$, while $I : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is the indicator function for the nonnegative reals, defined as

$$I(w) = \begin{cases} 0 & w \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Maximizing over the primal variables x and $\{z_b\}$, we obtain the dual function

$$g(\nu, \lambda) = \lambda^T q + \sup_x \left(U(x) - \nu^T x \right) + \sum_{b=1}^{B} \sup_{z_b \geq 0} \left((\nu_b c_b - \lambda)^T z_b \right)$$

The dual problem is then

$$\begin{array}{ll} \text{minimize} & g(\nu, \lambda) \\ \text{subject to} & \lambda \geq 0, \end{array} \tag{3}$$

with variables $\lambda \in \mathbf{R}^n$ and $\nu \in \mathbf{R}^B$. Note that this problem is also convex, as g is the pointwise supremum of a family of affine functions.

Duality. Denote the optimal value of the primal problem (2) by p^* and the optimal value of the dual problem (3) by d^*. Since all constraints are linear and the problem has nonempty feasible set, we have strong duality [Ber09, Prop. 5.3.1]:

$$p^* = d^*,$$

and there exists some dual variables λ^* and ν^* which achieve this bound.

Implicit constraints. Problem (3) has some additional implicit constraints. Observe that, if there exists some index i and buyer b such that $(\nu_b c_b - \lambda)_i > 0$, then

$$\sup_{z_b \geq 0} \left((\nu_b c_b - \lambda)^T z_b \right) \geq (\nu_b c_b - \lambda)^T (t e_i) = t (\nu_b c_b - \lambda)_i \to \infty,$$

as $t \uparrow \infty$. Here, we have chosen $z_b = te_i$, where e_i is the ith unit basis vector. This implies that the supremum is infinite if any element of $\nu_b c_b - \lambda$ is positive, so we must have

$$\nu_b c_b - \lambda \leq 0$$

as an implicit constraint for each $b = 1, \dots, B$. By a similar argument, we can show that if U is nondecreasing and $U(0) < \infty$, then we have the implicit constraint that

$$\nu \geq 0.$$

Adding both implicit constraints as explicit constraints gives the following constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & \sup_x \left(U(x) - \nu^T x \right) + \lambda^T q \\ \text{subject to} & \nu_b c_b \leq \lambda, \quad b = 1, \dots, B \\ & \lambda \geq 0, \ \nu \geq 0. \end{array} \tag{4}$$

Simplification. Note that the first constraint tells us that λ dominates $\nu_b c_b$ for each buyer $b = 1, \dots, B$. Using this, and the fact that $q \geq 0$, we can further simplify the dual problem:

$$\begin{array}{ll} \text{minimize} & \sup_x \left(U(x) - \nu^T x \right) + \sum_{i=1}^{n} q_i \max_b((\nu_b c_b)_i) \\ \text{subject to} & \nu \geq 0. \end{array} \tag{5}$$

After this simplification, the dual problem only has a positivity constraint on the dual variables and is relatively easy to solve with standard methods. This form also suggests that a tatonnement-style algorithm could be used to solve this problem, although we believe that solving the original problem (2) makes more sense in practice.

Dual variables. Both dual variables ν and λ have natural interpretations. To simplify the exposition, we will let U be differentiable in what follows, but note that this is not necessary and a suitable generalization exists via subgradients. From the supremum in the objective of (4) and (5), we have that the optimal dual variable $\nu^\star$ and optimal allocation $x^\star$ will satisfy

$$\nabla U(x^\star) = \nu^\star.$$

Thus, the dual variable $\nu^\star$ gives the marginal utilities for each buyer at the optimal allocations. Viewing these dual variables as prices, we interpret the objective in (5) as maximizing the net utility (utility minus cost) of the buyers, plus the seller's revenue from each item. (Recall that $(c_b)_i = 1$ if buyer b would accept item i with properties p_i and 0 otherwise.) While we may view an optimal dual variable $\nu^\star$ as the marginal prices for each buyer, we may view an optimal dual variable $\lambda^\star$ as the marginal price for each item. In particular, we know that at optimality, for the same reason we may rewrite (4) as (5), we have

$$\lambda_i^\star = \max_b((\nu_b c_b)_i), \tag{6}$$

for each $i = 1, \dots, n$. (This is an optimal point if $q \geq 0$, but not necessarily uniquely so.) Since the buyers themselves are partially sorted by preferences (from (1)), this observation says that that, over all buyers b willing to buy item i, $\lambda_i^\star$ is the maximum marginal price that any of these buyers is willing to pay for this item. In turn, this implies that the marginal price for each item i respects the partial ordering of properties in the sense that

$$\lambda_i^\star \geq \lambda_{i'}^\star \quad \text{if} \quad p_i \geq p_{i'}.$$

In English: if any item i has properties at least as good as another item i', then the marginal price of i is at least as large as that of i'. (This makes sense; any buyer of item i' would always prefer to buy item i for any price below that of i', since item i is at least as good.) Note that this relationship does not say anything about the price of items that are incomparable.

Implementation and payments. We can implement this mechanism by asking users to specify their preferences U and then solving the allocation problem (2) for the optimal allocation $x^\star$ and dual variable $\nu^\star$. We can charge each user i their marginal utility $\nu_i^\star$ for their allocation. In this case, this mechanism is essentially a first-price auction over an infinite set of goods. As in a standard first-price auction, we would expect users to shade their preferences in order to maximize their net utility (utility minus cost). In the following section, we take advantage of the convex nature of this market clearing problem to construct and incentive compatible payment rule.

4.2 A DSIC mechanism

While duality provides market-clearing prices ν for each buyer, these prices are not necessarily incentive-compatible. Thus, the question remains: if we clear the market by solving (2), what do we charge each buyer for their allocation? We answer this question by viewing the market clearing problem as a system welfare computation. This viewpoint leads to an 'auction' allocation rule and the associated dominant strategy incentive compatible (DSIC) payment rule.

Max-of-sum welfare. In this construction, we will have some compact (under whatever topology) space of allocations S and some locally-convex topological vector spaces $\mathcal{F}_1, \dots, \mathcal{F}_B$ of upper semicontinuous functions (with respect to the topology of S) mapping $S \to \mathbf{R}$, potentially with some additional restrictions such as concavity or monotonicity. We will define the following family, indexed by the set S and functional spaces $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_B$, of welfare functions $F : \mathcal{F} \to \mathbf{R}$ of the form:

$$F(f) = \sup_{x \in S} \left(\sum_{b=1}^{B} f_b(x) \right), \tag{7}$$

where $f \in \mathcal{F}$. We call these types of functions F *max-of-sum* due to their form. In this case, we can interpret the function $f_b \in \mathcal{F}_b$ as the *bid* for buyer b (who gets to construct any function they wish, as in (2)) and S as the set of possible allocations (the constraints of (2). If all buyers truthfully report their bid, this F would correspond exactly to the welfare gained by buyers and any solution $x^\star \in S$ (which exists by compactness of S and upper-semicontinuity of the $\{f_i\}$) to the problem is a welfare-maximizing allocation. It is not, of course, a-priori obvious that this will be the case, or how much the mechanism should charge users in order to ensure truthful bidding. We deal with these issues next.

Allocation rules. We define an *allocation rule* as a probability measure μ over (a sigma algebra of subsets of) the set S, for any fixed vector of bids f. (In our particular case, we will have $S \subseteq \mathbf{R}^n$, which means that the usual sigma-algebra over $\mathbf{R}^n$ works, but this construction holds more generally.) Any allocation is a sample drawn from the distribution μ. In our particular case, the allocation rule will be very simple: set $\mu(\{x^\star\}) = 1$, where $x^\star$ is any solution to the right hand side of (7). In other words, we set the allocation to always be a point mass on any solution of the right hand side of (7). Note that any probability distribution with support over the optimal points would also work. These rules might be of interest to ensure certain types of 'fairness' when the solution is not unique.

Myerson's lemma. Myerson's lemma [Mye81] gives us a way of constructing DSIC payment rules for allocation rules that are monotone in the finite-dimensional setting (*i.e.*, when $\mathcal{F}$ is itself a finite dimensional vector space over $\mathbf{R}$). Unfortunately, it is not clear what the correct generalization is when the space of possible bids $\mathcal{F}$ need not be finite-dimensional, as is true in the partially ordered market construction of §3. We will explicitly construct a payment rule and then show that it is DSIC under quasilinear utilities, although the result holds in more general settings.

Payment rule. The payment $P_b(f)$ that buyer b must make for a given set of bids $f \in \mathcal{F}$ and allocation rule (distribution) μ, is

$$P_b(f) = F(f_{-b}) - \sum_{b' \neq b} \int_S f_{b'}\, d\mu,$$

where $f_{-b} = (f_1, \dots, f_{b-1}, 0, f_{b+1}, \dots, f_B) \in \mathcal{F}$. In the case that we choose the allocation rule μ to always be a point mass at the optimal point $x^\star$ which maximizes (7) over the bids f, this simplifies to

$$P_b(f) = F(f_{-b}) - \sum_{b' \neq b} f_{b'}(x^\star). \tag{8}$$

Note that this payment rule can be evaluated by solving the market clearing problem (2) $B+1$ times: once to compute an optimal allocation $x^\star$, and B times to compute the allocation without each of the B buyers.

Interpretation. The interpretation of this rule is particularly simple: we charge buyer b the marginal utility lost by all other buyers for having included buyer b's bid. In other words, buyer b must pay their externality. This payment rule uses essentially the same idea as the Vickrey auction, except in this more general case. We can recover the Vickrey auction payment and allocation rules directly from this construction as the special case where $S = \{e_b \mid b = 1, \dots, B\}$ where e_b is the bth unit basis vector and the functions f_b are linear and depend only on the bth component; *i.e.*, when $f_b : S \to \mathbf{R}$ and the family of possible functions are those for which

$$\mathcal{F}_b = \{f_b \mid f_b(x) = \alpha_b x_b \text{ for some } \alpha_b \in \mathbf{R}_+\},$$

for each $b = 1, \dots, B$. For the Vickrey auction with a single item, the set S simply says that the item must be allocated to one buyer and problem (7) says that the welfare is the allocation which takes the maximum over all bids. The payment rule (8) then says that the winning buyer (ties broken based on the definition of μ) pays the second-highest bid and the rest pay nothing. We leave extensions of this model to handle reserve prices, indivisible goods, and other variants considered in the literature to future work. We also note that the usual issues with sybils in VCG mechanisms also apply here (false-name bids) [YSM04]. We leave exploration of false-name-proof mechanisms to future work.

Proof of DSIC. The proof that this mechanism is dominant strategy incentive compatible is mostly notation-chasing. Let $\bar{f}_b : \mathcal{F}_b \to \mathbf{R}$ be any bid for buyer b and let $f_b : \mathcal{F}_b \to \mathbf{R}$ be the true valuation; *i.e.*, for any allocation μ, the (expected) value to player b of this allocation is $\int_S f_b \, d\mu$. Then, by definition, f_{-b} does not contain player b's bid (given by $\bar{f}_b$). We will show that the payoff for player b having bid f_b is always at least as large as the payoff for her having bid any other $\bar{f}_b$. In particular, consider the difference in payoffs:

$$\underbrace{\left(\int_S f_b \, d\mu - \left(F(f_{-b}) - \sum_{b' \neq b} \int_\mu f_{b'} \, d\mu\right)\right)}_{\text{payoff for bidding } f_b} - \underbrace{\left(\int_S f_b \, d\bar{\mu} - \left(F(f_{-b}) - \sum_{b' \neq b} \int_\mu f_{b'} \, d\bar{\mu}\right)\right)}_{\text{payoff for bidding } \bar{f}_b}.$$

Here μ denotes any allocation rule consistent with our definition (that is, any probability distribution over maximizers of the right hand side of (7)) for the bids $(f_1, \dots, f_B)$ and $\bar{\mu}$ is the same but for the bids $(f_1, \dots, f_{b-1}, \bar{f}_b, f_{b+1}, \dots f_B)$. Rearranging gives that this expression is equal to

$$\sum_{b=1}^{B} \int_S f_b \, d\mu - \sum_{b=1}^{B} \int_S f_b \, d\bar{\mu} = F(f) - \int_S \left(\sum_{b=1}^{B} f_b\right) d\bar{\mu} \geq 0.$$

Where the inequality follows from the fact that any expectation is never larger than the supremum, along with the definition of F.

5 Examples

We conclude this paper with a few examples. First, we show a simple numerical example to demonstrate the payment and allocation rules, and how prices change under different preferences. Next, we show how to model decentralized lending protocols, which offer a variety of different yield products, in our framework. Finally, we model restaking networks, in which the utility for any buyer depends not only on their allocation but also on the actions of all other buyers. We show that this situation fits into our market clearing mechanism §3 and conjecture that the DSIC payment rule developed in §4 extends as well.

5.1 Simple example

Consider a market with $B = 2$ buyers and $n = 3$ yield bearing assets with different ratings and yields, depicted in figure 3. There is one (divisible) unit of each item. We consider a few

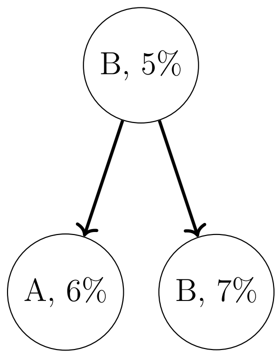

Figure 3: Three yield bearing assets with various ratings, and their associated partial ordering.

different scenarios and look at the market clearing prices (dual variables) and the payment rules in each case.

Homogenous market. First, consider the case where each buyer would accept any item. Buyers express utilities in units of yield. The resulting (identical) preference vectors are

$$c_1 = c_2 = (6, 5, 7)$$

where assets are indexed via a post-order traversal of the graph in figure 3. We assume that both buyers have a square root utility function

$$u_b(x_b) = \sqrt{x_b}.$$

Solving the market clearing problem (2) results in symmetric allocation and payments:

$$x_1^\star = x_2^\star = 9, \qquad \text{and} \qquad z_1^\star = z_2^\star = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

The dual variables are $\nu_1^\star = \nu_2^\star = 1/6$, and we can easily verify that (6) holds:

$$\nu_1^\star \cdot c_1 = \nu_2^\star \cdot c_2 = \frac{1}{6} \cdot \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix} = \lambda^\star = \begin{bmatrix} 1 \\ 5/6 \\ 7/6 \end{bmatrix}.$$

Each buyer pays 1.24 resulting in a net utility of 1.76.

Rating preferences. Next, we consider the case where buyer 1 only wants an A-rated asset, but buyer 2 is indifferent. The following vectors encode these preferences:

$$c_1 = (6, 0, 0)$$
$$c_2 = (6, 5, 7)$$

Both buyers again have a square root utility function. Solving the market clearing problem (2) results in the allocation

$$x_1^\star = 6, \quad z_1^\star = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad x_2^\star = 12, \quad z_1^\star = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The dual variables are $\nu_1^\star = 0.204$. $\nu_2^\star = 0.144$, and we can easily verify that (6) holds:

$$\nu_1^\star \cdot c_1 = \begin{bmatrix} 1.224 \\ 0 \\ 0 \end{bmatrix} \leq \lambda^\star = \begin{bmatrix} 1.225 \\ 0.722 \\ 1.01 \end{bmatrix} \qquad \text{and} \qquad \nu_2^\star \cdot c_2 = \begin{bmatrix} 0.864 \\ 0.72 \\ 1.01 \end{bmatrix} \leq \lambda^\star = \begin{bmatrix} 1.225 \\ 0.722 \\ 1.01 \end{bmatrix},$$

with each elementwise inequality saturated for at least one buyer. In this case, buyer 1 pays 0.78 but buyer 2 pays nothing. This payment makes sense; buyer 2 was only allocated items that buyer 1 would not accept. The resulting net utilities are 1.67 and 3.47 respectively.

Different utilities. Finally, we consider the case where the buyers have different utility functions but both are willing to accept either A- or B-rated assets. Here, we use the utility functions

$$u_1(x_1) = \log(x_1 + 1) \qquad \text{and} \qquad u_2(x_2) = \sqrt{x_2}.$$

Solving the market clearing problem (2) results in the allocation

$$x_1^\star = 5.94, \quad z_1^\star = \begin{bmatrix} 0.34 \\ 0.35 \\ 0.31 \end{bmatrix} \qquad \text{and} \qquad x_2^\star = 12.06, \quad z_1^\star = \begin{bmatrix} 0.66 \\ 0.65 \\ 0.67 \end{bmatrix}.$$

The dual variables are $\nu_1^\star = 0.144$. $\nu_2^\star = 0.144$, and we can easily verify that (6) holds:

$$\nu_1^\star \cdot c_1 = \begin{bmatrix} 0.864 \\ 0.72 \\ 1.01 \end{bmatrix} \leq \lambda^\star = \begin{bmatrix} 0.864 \\ 0.72 \\ 1.008 \end{bmatrix} \qquad \text{and} \qquad \nu_2^\star \cdot c_2 = \begin{bmatrix} 0.864 \\ 0.72 \\ 1.01 \end{bmatrix} \leq \lambda^\star = \begin{bmatrix} 0.864 \\ 0.72 \\ 1.008 \end{bmatrix},$$

This result makes sense; the buyers have different utility functions, but the market clearing mechanism chose an allocation such that their marginal utilities match. In this case, buyer 1 pays 0.77 and buyer 2 pays 1.01. The resulting net utilities are 5.17 and 2.47 respectively.

5.2 Decentralized lending protocols

On-chain lending markets match suppliers, who want to earn yield on their existing holdings, to users who want to borrow against their existing portfolio. These protocols are overcollateralized: borrowers have to deposit collateral greater than the initial value of their loan to open a position. Such loans, which are similar to securities-based lending and Lombard loans, allow a user who holds a large quantity of a risky asset (such as Ethereum) to borrow numéraire assets (such as the dollar-pegged USDC) to invest elsewhere. These markets have grown to over $35 billion in supplied assets and have had at least $2 billion in daily outstanding loans since 2021 [Sta25a].

Classic DeFi lending protocols. The first on-chain lending protocols within decentralized finance (DeFi) include Compound [Kao+20] and Aave [Fu+21; SSS22]. These protocols pool together suppliers' assets and algorithmically set borrow interest rates. Losses from borrowers who default are shared amongst the pool, reducing the variance in returns for suppliers. However, these protocols have a fixed interest rate model and cannot dynamically adjust to changing market conditions or different borrower profiles. As a result, these protocols must be conservative in choosing loan parameters to mitigate default risk.

Curatorial lending protocols. Newer lending protocols, such as Morpho [Fra23] and Euler [Tea24b], introduce a third-party known as a *curator*. In these markets, curators accumulate assets from suppliers into 'vaults' which each target a particular borrower profile and, as a result, don't need such conservative risk parameters. Competition amongst curators results in dynamic price discrimination on the rates charged to the different markets [Ber+24]. These curatorial markets have grown to over $6 billion in assets in 2025 [Sta25c].

Market clearing model. Consider a curatorial market with V vaults (items) and B borrowers (buyers). Loans from each vault $v = 1, \dots, V$ can only be used for particular, specified actions. Each borrower $b = 1, \dots, B$, set $(c_b)_v = 1$ if they can use vault v for their desired actions and $(c_b)_v = 0$ otherwise. If a vault v has supply $s_v > 0$ current demand $d_v \geq 0$, then for a loan of size x the vault charges an interest rate

$$\kappa_v \cdot \frac{d_v + x}{s_v},$$

where $\kappa_v > 0$ is a constant. Borrower b earns revenue $r_b(x_b)$, which is a concave, nondecreasing function of the loan size x_b. Thus, after loans are allocated, this buyer has net utility

$$u_b(x_b) = r_b(x_b) - \kappa_v \cdot \frac{d_v + x_b}{s_v} \cdot x_b.$$

The market clearing problem aims to allocate the supply to borrowers that minimizes their borrowing cost while honoring vault constraints.

Discussion. We could clear the other side of this market—allocation from asset suppliers to curated vaults—with our market clearing mechanism as well. Each asset supplier attempts to maximize their revenue while minimizing risk from bad loans, expressed via preferences over the vaults to which they would supply assets. Since the mechanism only requires specification of utility functions and the preference vector, a protocol could also dynamically rebalance vaults on behalf of the suppliers. Also note that, in practice, the interest rate typically depends on not only an individual borrower's loan but also the loans allocated to all other borrowers at the same time. Modeling this results in a non-separable utility function, similar to that of our next example.

5.3 Restaking networks

Restaking markets in proof-of-stake cryptocurrency networks exhibit similar behavior. These markets allows asset holders to lend out their assets for additional yield.

Staking. Proof-of-stake cryptocurrency networks provide economic incentives for users, called *stakers*, to validate the state of the network. Specifically, these users earn rewards for checking the validity of blocks submitted to the network. To earn these rewards, stakers must lock up assets as collateral. If a staker incorrectly performs their validation duties, they are financially penalized ('slashed', in industry parlance). See [Sal21] for additional details.

Restaking. Restaking allows for a staker to earn rewards for validating multiple networks ('services', in industry parlance) [Tea24a] using the same assets. In other words, a single staker pledges their assets as collateral to multiple services. These stakers now have many different validation duties—some from each service they validate—and, as a result, may be penalized by any of these services, raising the risk of default and of liquidation cascades. Durvasula and Roughgarden [DR24] first formalized the excess default risk, and follow-up work by Chitra and Pai [CP24] analyzed the impact of how incentives paid affected this default risk. The restaking service Eigenlayer on the Ethereum blockchain has over \$15 billion of assets [Sta25b].

Market clearing model. We view each service's reward (yield) as an item and each staker as a buyer. Each buyer aims to maximize expected revenue within their risk parameters. They will evaluate services based on a number of properties such as historical economic performance, penalty (slashing) rates, service uptime, current yield, and service credit risk. In practice, these properties may be collapsed into only rating and yield, as in §5.1. We note that restaking allows for a single staker to delegate their stake to multiple services. We adapt the incentivized restaking model of [CP24]. Each service $s = 1, \dots, S$ costs c_s and pays rewards r_s, split pro-rata amongst all its validators. We consider SB items, one for each service-validator combination. The preferences can be encoded as

$$(c_b)_s = \begin{cases} 1 & \text{staker } b \text{ accepts service } s \\ 0 & \text{otherwise,} \end{cases}$$

where c_b only 'hits' the relevant items for validator b. The (nonseparable, concave) utility is then

$$U(x) = \sum_{s=1}^{S} \left(r_s \cdot \frac{x_{b,s}}{\sum_{b=1}^{B} x_{b,s}} - c_s \cdot \sum_{b=1}^{B} x_{b,s} \right).$$

We can similarly clear this market by solving (2). Note that, while the objective is not separable, we conjecture that can use the same ideas outlined in §4.2: charging buyer b her externality results in a DSIC payment rule. We leave full characterization to future work.

6 Conclusion

In this paper, we constructed markets over semi-fungible (or partially-ordered) assets. Such markets take into account this partial ordering in order to provide additional liquidity for buyers and sellers: any buyer who wishes to buy an item is, if they receive an item, guaranteed an item at least as good as the requested one. Thus, any seller has 'more chances' to sell their items. We then showed that clearing these markets is a convex optimization problem that can be efficiently solved on modern hardware. The dual problem admits a nice interpretation in terms of the market clearing prices, and we showed that these prices respect the partial order; *i.e.*, items that are 'better' with respect to this order have a price no lower than those which are 'worse'. Finally, we constructed a dominant strategy incentive compatible mechanism such that buyers are incentivized to bid their true valuations, which is efficient to implement in practice. We believe that our mechanism opens many avenues for interesting future work.

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References

[AD03] Yacine Aıt-Sahalia and Jefferson Duarte. "Nonparametric option pricing under shape restrictions". In: *Journal of Econometrics* 116.1-2 (2003), pp. 9–47.

[AR22] Eleni Aristodemou and Adam M Rosen. "A discrete choice model for partially ordered alternatives". In: *Quantitative Economics* 13.3 (2022), pp. 863–906.

[BCS15] Eric Budish, Peter Cramton, and John Shim. "The high-frequency trading arms race: Frequent batch auctions as a market design response". In: *The Quarterly Journal of Economics* 130.4 (2015), pp. 1547–1621.

[Ber+24] Charles Bertucci et al. "Agents' Behavior and Interest Rate Model Optimization in DeFi Lending". In: *Available at SSRN 4802776* (2024).

[Ber09] Dimitri Bertsekas. *Convex Optimization Theory*. Vol. 1. Athena Scientific, 2009.

[BM93] Shelby L Brumelle and Jeffrey I McGill. "Airline seat allocation with multiple nested fare classes". In: *Operations research* 41.1 (1993), pp. 127–137.

[Bud+23] Eric Budish et al. *Flow trading*. Tech. rep. National Bureau of Economic Research, 2023.

[CG11] Ning Chen and Arpita Ghosh. "A market clearing solution for social lending". In: *IJCAI*. Vol. 11. 2011, pp. 152–157.

[CP24] Tarun Chitra and Mallesh Pai. "How much should you pay for restaking security?" In: *arXiv preprint arXiv:2408.00928* (2024).

[CV07] Bruno Codenotti and Kasturi Varadarajan. "Computation of market equilibria by convex programming". In: *Algorithmic Game Theory* (2007), pp. 135–158.

[DAE24] Theo Diamandis, Guillermo Angeris, and Alan Edelman. "Convex Network Flows". In: *arXiv preprint arXiv:2404.00765* (2024).

[DR24] Naveen Durvasula and Tim Roughgarden. "Robust Restaking Networks". In: *arXiv preprint arXiv:2407.21785* (2024).

[EG59] Edmund Eisenberg and David Gale. "Consensus of subjective probabilities: The pari-mutuel method". In: *The Annals of Mathematical Statistics* 30.1 (1959), pp. 165–168.

[Fra23] Paul Frambot. "Introducing MetaMorpho: Permissionless lending vaults on Morpho Blue". In: (Nov. 2023). URL: `https://morpho.mirror.xyz/vPkSBEZvIoVDfx-GBOsTOvjI14bqul-PA-LMe3sBlBw`.

[Fu+21] Watson Fu et al. "Aave Market Risk Assessment". In: (2021). `https://gauntlet.network/reports/aave`.

[GS99] Faruk Gul and Ennio Stacchetti. "Walrasian equilibrium with gross substitutes". In: *Journal of Economic theory* 87.1 (1999), pp. 95–124.

[JLT97] Robert A Jarrow, David Lando, and Stuart M Turnbull. "A Markov model for the term structure of credit risk spreads". In: *The review of financial studies* 10.2 (1997), pp. 481–523.

[Kao+20] Hsien-Tang Kao et al. "An analysis of the market risk to participants in the compound protocol". In: *Third international symposium on foundations and applications of blockchains*. Vol. 362. 2020.

[KL17] Albert S Kyle and Jeongmin Lee. "Toward a fully continuous exchange". In: *Oxford Review of Economic Policy* 33.4 (2017), pp. 650–675.

[Lev24] Matt Levine. "You Want Some Stocks That Go Down". In: *Bloomberg Opinion* (Oct. 2024). Accessed: 2024-12-04. URL: `https://www.bloomberg.com/opinion/articles/2024-10-28/you-want-some-stocks-that-go-down`.

[LR08] Francis A Longstaff and Arvind Rajan. "An empirical analysis of the pricing of collateralized debt obligations". In: *The Journal of Finance* 63.2 (2008), pp. 529–563.

[Mye81] Roger B Myerson. "Optimal auction design". In: *Mathematics of operations research* 6.1 (1981), pp. 58–73.

[New24] Bloomberg News. "It's PhDs Against MBAs in Credit Investing as Quants Jump In". In: *Bloomberg* (May 2024). Accessed: 2024-12-04. URL: `https://www.bloomberg.com/news/articles/2024-05-20/it-s-phds-against-mbas-in-credit-investing-as-quants-jump-in`.

[Sal21] Fahad Saleh. "Blockchain without waste: Proof-of-stake". In: *The Review of financial studies* 34.3 (2021), pp. 1156–1190.

[SSS22] Xiaotong Sun, Charalampos Stasinakis, and Georgios Sermpinis. "Liquidity risks in lending protocols: Evidence from aave protocol". In: *arXiv preprint arXiv:2206.11973* (2022).

[Sta25a] DeFi Llama Staff. *Aave*. Feb. 2025. URL: `https://defillama.com/protocol/aave`.

[Sta25b] DeFi Llama Staff. *Eigenlayer*. Feb. 2025. URL: `https://defillama.com/protocol/eigenlayer`.

[Sta25c] DeFi Llama Staff. *Morpho*. Feb. 2025. URL: `https://defillama.com/protocol/morpho`.

[Tea24a] EigenLayer Team. "Eigenlayer: The restaking collective". In: *White paper* (2024).

[Tea24b] Euler Team. "Euler Whitepaper". In: (May 2024).

[Vaz07] Vijay V Vazirani. "Combinatorial algorithms for market equilibria". In: *Algorithmic game theory* (2007), pp. 103–134.

[Vég14] László A Végh. "Concave generalized flows with applications to market equilibria". In: *Mathematics of Operations Research* 39.2 (2014), pp. 573–596.

[Whi24] Christopher Whittall. "Portfolio trading surges and heads multi-asset". In: *International Financing Review (IFR)* (Oct. 2024). URL: `https://www.ifre.com/story/4872923/portfolio-trading-surges-and-heads-multi-asset-mlckjj4vsd`.

[YSM04] Makoto Yokoo, Yuko Sakurai, and Shigeo Matsubara. "The effect of false-name bids in combinatorial auctions: New fraud in Internet auctions". In: *Games and Economic Behavior* 46.1 (2004), pp. 174–188.