An Efficient Algorithm for Optimal Routing Through Constant Function Market Makers

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tl;dr: It's all convex optimization

Routing (multi-DEX swaps, etc.) is a convex[1] optimization problem, so it can be *very efficiently* solved to *verifiable* global optimality.

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This talk: the 'very efficiently' part

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Outline

Review: Constant Function Market Makers

- Most DEXs are implemented as *constant function market makers* (CFMMs)
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- Maps reserves $R \in \mathbb{R}^n_+$ to a real number
- Is concave and increasing
- Accepts trade $\Delta \to \Lambda$ if $\varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R)$.

Most DEXs are CFMMs

- Geometric mean trading function (Balancer, Uniswap, etc...):

$$\varphi(R) = \left(\prod_{i=1}^{n} R_i^{w_i} \right)^{1/n}$$

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- Curve:

$$\varphi(R) = \mathbf{1}^T R - \alpha \prod_{i=1}^{n} R_i^{-1}$$

where $\alpha > 0$.

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- **Problem:** How to split trade? **Solution:** build a router

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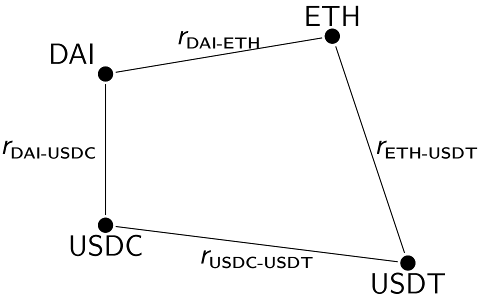
Networks of CFMMs

- Common representation: undirected graph with exchange rates



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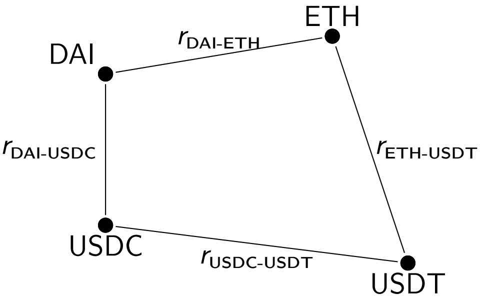
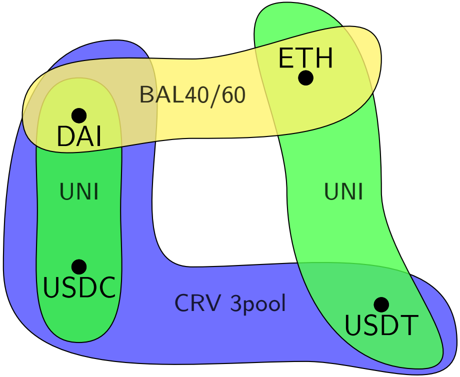


- But how to handle three pools? Multiple CFMMs?

Networks of CFMMs

- The token-CFMM network is a hypergraph: edges can connect more than 2 vertices



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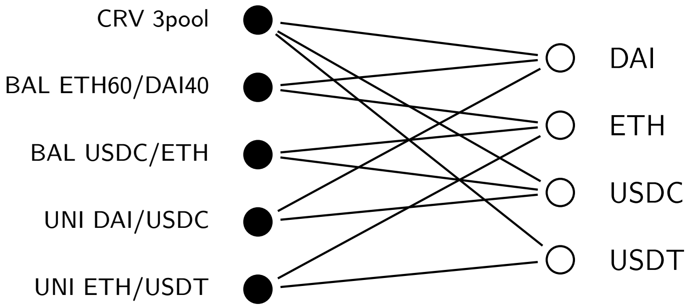

- Good bookkeeping is essential!

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- Trade accepted if $\varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i) \geq \varphi_i(R_i)$

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- Matrices A_i map token's *local* index in CFMM i to global index, *e.g.*, ,

Token	Local Index	Global Index
DAI	1	3
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- The overall net trade with the network is

$$\Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i)$$

Simplifying the Model

- We ignore gas fees
- We don't worry about transaction execution ordering
- We can return to these later...

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- Each individual CFMM is defined by trading constraints

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- Utility function U gives our satisfaction with the net trade
- We can also use U to encode constraints
- **Arbitrage:** Find the most profitable nonnegative net trade

$$U(\Psi) = c^T \Psi - \mathbb{I}(\Psi \geq 0)$$

  - The vector c is a positive price vector
  - Indicator function $\mathbb{I}(\Psi \geq 0) = 0$ if $\Psi \geq 0$ and $+\infty$ otherwise

Swaps: trade token i for j

- Goal: maximize output of token j given fixed input of token i
- Constraints: input exactly Δ^i of token i and only get token j

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- More generally, we can optimally purchase or liquidate a basket of tokens
- Capturing "arbitrage" opportunities as part of the swap

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- Given these prices, you can arbitrage each CFMM independently & in parallel
- Strong duality $\implies$ dual problem has the same optimal value
- Strong duality $\implies$ certificate of optimality (very cheap to check)

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$$\begin{array}{ll} \text{maximize} & (A_i^T \nu)^T (\Lambda_i - \Delta_i) \\ \text{subject to} & \varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i) \geq \varphi_i(R_i) \\ & \Delta_i \geq 0, \quad \Lambda_i \geq 0 \end{array}$$

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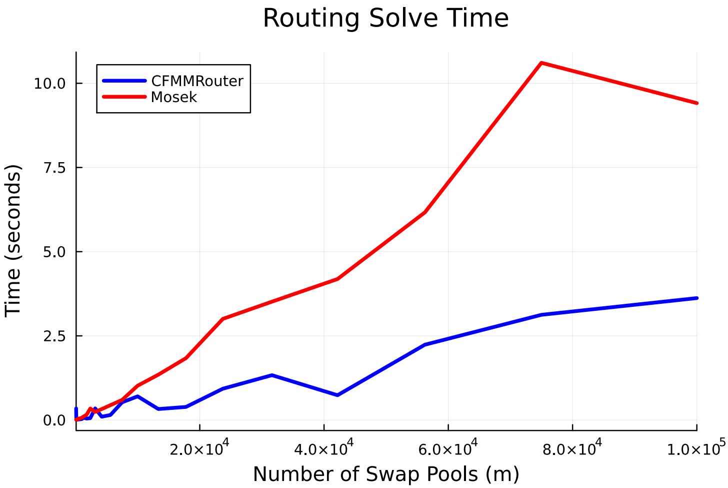
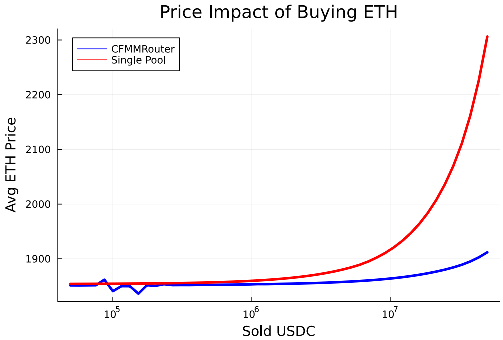
- To add a DEX, **only need to define this arbitrage function**

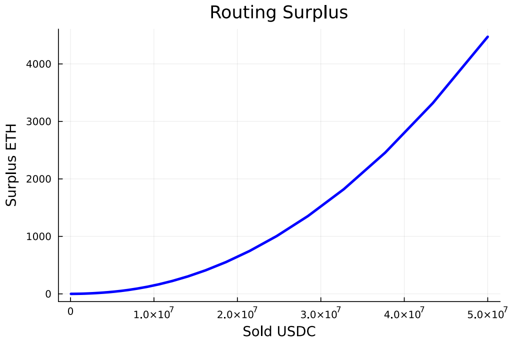
Outline

Our solver `CFMMRouter` is faster than commercial convex solvers

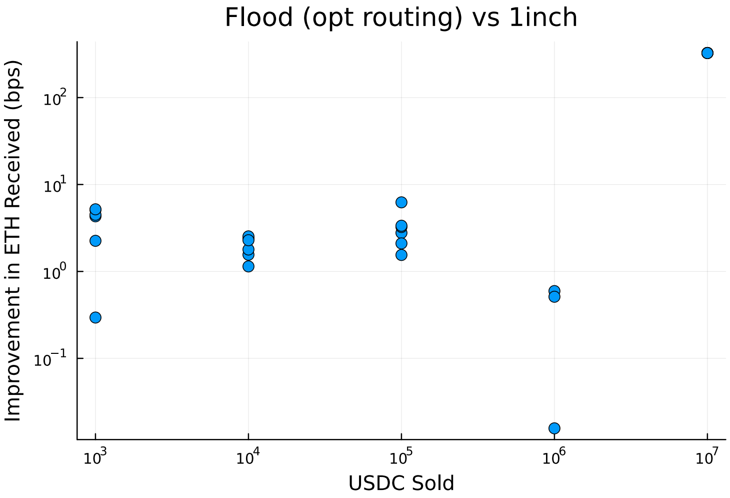


We see way less price impact for large txns





And it beats 1inch in production on Arbitrum (flood.bid)



Routing package on Github: `CFMMRouter.jl`

Flood in beta on Arbitrum: `flood.bid`

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- And we can prove a feasible point is optimal
- We construct an efficient algorithm using convex duality
- This algorithm is implemented in `CFMMRouter.jl`

Future work includes expanding this framework

- Routing with gas fees (nonconvex—need good heuristics)
- Routing through liquidations
- Routing with probabilistic constraints when TXs may fail (*e.g.*, cross-chain)

For more info, check out our paper & **`CFMMRouter.jl`**

Paper

Thank you!

Theo Diamandis

`tdiamand@mit.edu`

Appendix

Optimality conditions

For the primal problem

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The optimality conditions are

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\lambda_i \gamma_i \nabla \varphi_i(R_i + \gamma_i \Delta_i^\star - \Lambda_i^\star) \leq A_i^T \nu^\star \leq \lambda_i \nabla \varphi_i(R_I + \gamma_i \Delta_i^\star - \Lambda_i^\star), \qquad i = 1, \ldots, m
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- Gas cost for CFMM i is q_i
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- ...but we have good heuristics for this type of problem

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- This is relatively easy: simple algorithm & closed form solution within a tick
- **Answer 2:** The φ constraint is a bit of a lie...
- Only need a convex reachable reserve set (or, equivalently, trading set):

$$\varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R) \iff R + \gamma\Delta - \Lambda \in S(R)$$